阿里巴巴全球竞赛决赛试题

一、代数与数论　　Algebra&Number Theory

1.

设 F 为域。考虑 F^n 上的如下环结构，加法是通常的向量加法，乘法定义为

$$(x_1,\ldots,x_n)\cdot(y_1,\ldots,y_n)=(x_1y_1,\ldots,x_ny_n).$$

设 $\Lambda\subset F^n$ 为包含 $(1,\ldots,1)$ 的子环。假设 Λ 为整环，而且它作为加法群是有限生成的。试证对于 Λ 的任何非零元 $(x_1,\ldots,x_n)$，我们有 $\prod_{i=1}^n x_i\neq 0$。

Let F be a field. Consider the ring structure on F^n where addition is the usual vector addition and multiplication is defined by

$$(x_1,\ldots,x_n)\cdot(y_1,\ldots,y_n)=(x_1y_1,\ldots,x_ny_n).$$

Let $\Lambda\subset F^n$ be a subring containing $(1,\ldots,1)$. Suppose Λ is an integral domain, and its underlying additive group is finitely generated. Prove that for every nonzero element $(x_1,\ldots,x_n)$ in Λ, one has $\prod_{i=1}^n x_i\neq 0$.

2.

设群 G 作用在集合 Ω 上，使得所有 G-轨道都是无限集。设 Γ,Δ 为 Ω 的有限子集。试证存在 $g\in G$ 使得 $g\Gamma\cap\Delta=\varnothing$。

Let a group G act on a set Ω such that all G-orbits are infinite. Let Γ,Δ be finite subsets of Ω. Prove that there exists $g\in G$ with $g\Gamma\cap\Delta=\varnothing$.

3.

设 V 为域 F 上的有限维向量空间，$\mathrm{char}(F) \neq 2$，令 $q : V \to F$ 为二次型，也

就是说：存在对称 F-双线性型 $B : V^2 \to F$ (必然唯一)使得 $q(v) = B(v,v)$ 对所有 v 成立。对所有域扩张 $F \hookrightarrow E$，定义二次型 q 到 E 上的基变换为

$$q_E : E \otimes_F V \to E; \quad q_E(a \otimes v) = a^2 q(v), \quad a \in E, v \in V.$$

我们说 q 是迷向的，如果 $v \neq 0 \iff q(v) \neq 0$。

(a) 试证如果 q 迷向，而 $[E : F]$ 是奇数，那么 q_E 也是迷向的。

(b) 以上叙述在 $[E : F]$ 为偶数时是否成立？

Let V be a finite-dimensional vector space over a field F with $\mathrm{char}(F) \neq$

2, and let $q : V \to F$ be a quadratic form, which means: there is a symmetric F-bilinear form $B : V \times V \to F$ (necessarily unique) such that $q(v) = B(v,v)$ for all v. For any field extension $F \hookrightarrow E$, we define the base-change quadratic form of q to E by

$$q_E : E \otimes_F V \to E; \quad q_E(a \otimes v) = a^2 q(v), \quad a \in E, v \in V.$$

We say q is *anisotropic* if $v \neq 0 \iff q(v) \neq 0$.

(a) Show that if q is anisotropic and $[E : F]$ is an odd positive integer, then q_E is also anisotropic.

(b) Does the above statement still hold if $[E : F]$ is even?

4.

找出所有满足以下条件的有限群 G：

- G 的阶是相异素数的积，换句话说，存在相异素数 $p_1, \ldots, p_m$ 使得 $\#G = p_1 \cdots p_m$；
- G 的所有非平凡元都是素数阶的，换句话说每个元素的阶数都属于 $\{1, p_1, \ldots, p_m\}$。

(注记：答案和 m 有关；例如当 $m = 2$ 时存在许多这样的 G；您必须对它们分类。)

Find all finite groups G satisfying the following conditions:

- the order of G is the product of distinct primes, i.e. $\#G = p_1 \cdots p_m$ for some distinct primes $p_1, \ldots, p_m$; and
- all non-trivial elements of G have prime order, that is, the order of every element belongs to $\{1, p_1, \ldots, p_m\}$.

(Note: The answer depends on m; for example, when $m = 2$, there are many such G; you need to classify them.)

5.

找出所有 (k,α),其中 $k>2$ 是整数而 $\alpha\neq 0$ 是复数,使得

$$\alpha\in\left\{re^{\frac{i\pi}{k}}\mid r\in\mathbb{R}\right\},\quad \alpha+\alpha^{-1}\in\left\{m+n\sqrt{-2}\mid m,n\in\mathbb{Z}\right\}.$$

Find all pairs (k,α), where $k>2$ is an integer and $\alpha\neq 0$ is a complex number, such that

$$\alpha\in\left\{re^{\frac{i\pi}{k}}\mid r\in\mathbb{R}\right\},\quad \alpha+\alpha^{-1}\in\left\{m+n\sqrt{-2}\mid m,n\in\mathbb{Z}\right\}.$$

二、分析与方程 Analysis&Differential Equations

1.

假设$g(x)$是定义在$\mathbb{R}^3$上的光滑Schwarz函数（也称速降函数），满足条件

$$\int_{|y|=1} g(x+y)d\sigma(y)=0,\quad \forall x\in\mathbb{R}^3.$$

这里$d\sigma(y)$是球面$\{|y|=1\}$上的标准面积单元。证明$g=0.$

Assume that $g(x)$ is a Schwartz function on $\mathbb{R}^3$ satisfying that

$$\int_{|y|=1} g(x+y)d\sigma(y)=0,\quad \forall x\in\mathbb{R}^3.$$

Here $d\sigma(y)$ is the surface measure on the sphere $\{|y|=1\}$. Prove that $g=0$.

2.

假设$\mathbb{R}^3$上的球对称函数$u(x)$(也就是当$|x|=|y|$时有$u(x)=u(y)$)满足方程

$$\Delta u - u + |u|^2 u = 0, \quad \forall x \in \mathbb{R}^3.$$

如果$u \in C^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$，证明存在常数$C$使得

$$|u(x)| \leq C e^{-\frac{1}{2}|x|}.$$

Let $u \in C^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ be a spherically symmetric function (that is $u(x) = u(y)$ whenever $|x| = |y|$) verifying the equation

$$\Delta u - u + |u|^2 u = 0, \quad \forall x \in \mathbb{R}^3.$$

Prove that there exists positive constant C such that

$$|u(x)| \leq C e^{-\frac{1}{2}|x|}.$$

3.

考虑$\mathbb{R}^n$中的有界区域Ω，以及定义在这个区域上的非负函数κ使得对常数$\alpha > 1$，$M > 0$, $E_0 > 0$有

$$M \leq \int_\Omega \kappa dx, \quad \int_\Omega \kappa^\alpha dx \leq E_0.$$

证明存在只依赖于M, E_0, α的常数C使得

$$\|v\|_{L^2(\Omega)} \leq C\left(\|\nabla v\|_{L^2(\Omega)} + \int_\Omega \kappa |v| dx \right), \quad \forall v \in H^1(\Omega).$$

Let Ω be a bounded domain in $\mathbb{R}^N$ and κ be a non-negative function such that

$$M \leq \int_\Omega \kappa dx, \quad \int_\Omega \kappa^\alpha dx \leq E_0$$

for some constants $\alpha > 1$, $M > 0$, $E_0 > 0$. Prove that there exists a constant C depending only on M, E_0, α such that

$$\|v\|_{L^2(\Omega)} \leq C\left(\|\nabla v\|_{L^2(\Omega)} + \int_\Omega \kappa |v| dx \right), \quad \forall v \in H^1(\Omega).$$

4.

在$\mathbb{R}^n$上考虑薛定谔方程

$$i\partial_t u + \Delta u = 0, \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n.$$

假设初始值$u_0 \in L^2(\mathbb{R}^n)$。证明

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{|x|\le\sqrt{t}} |u(t,x)|^2 dxdt = 0.$$

Consider the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n.$$

Assume that $u_0 \in L^2(\mathbb{R}^n)$. Prove that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{|x|\le\sqrt{t}} |u(t,x)|^2 dxdt = 0.$$

5.

考虑标准的2维环面$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$，以及上面的半径为0.1的圆圈$S$。证明存在常数$C > 0$使得对任意定义在环面上，并且满足方程

$$\partial_{x_1}^2 f - \partial_{x_2}^2 f = \lambda f, \quad \lambda \ne 0$$

的函数f均满足不等式

$$\|f\|_{L^2(S,\mathrm{d}s)} \le C\|f\|_{L^2(\mathbb{T}^2)}.$$

其中$\mathrm{d}s$为圆圈的弧长测度.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-dimensional torus and S be a fixed circle of radius 0.1 on $\mathbb{T}^2$ with the arc length measure $\mathrm{d}s$. Prove that there exists a constant $C > 0$ such that for any f on $\mathbb{T}^2$ satisfying

$$\partial_{x_1}^2 f - \partial_{x_2}^2 f = \lambda f, \quad \lambda \neq 0,$$

we have

$$\|f\|_{L^2(S,\mathrm{d}s)} \leq C\|f\|_{L^2(\mathbb{T}^2)}.$$

三、几何与拓扑 Geometry&Topology

1.

设 S_g 是亏格为 g 的可定向闭曲面, N_{2g} 是亏格为 $2g$ 的不可定向闭曲面 (即 N_{2g} 由球面粘接 $2g$ 个 “交叉帽” 得到). 设 $f : N_{2g} \to S_g$ 是连续映射. 证明：诱导映射 $f_* : H_2(N_{2g}; \mathbb{Z}/2\mathbb{Z}) \to H_2(S_g; \mathbb{Z}/2\mathbb{Z})$ 恒为 0.

Let S_g be a closed orientable surface of genus g, and let N_{2g} be a closed non-orientable surface of genus $2g$ (i.e. N_{2g} is obtained by attaching $2g$ cross-caps to a sphere). Let $f : N_{2g} \to S_g$ be a continuous map. Prove that the induced map $f_* : H_2(N_{2g}; \mathbb{Z}/2\mathbb{Z}) \to H_2(S_g; \mathbb{Z}/2\mathbb{Z})$ is zero.

2.

设 S^3 是 $\mathbb{R}^4$ 中的单位球面，赋予标准的李群结构，x 为 de Rham 上同调 $H^3(S^3, \mathbb{R})$ 的一个非零元. 证明：不存在李群同构 $f : S^3 \to S^3$ 使得 $f^*(x) = -x$.

Let S^3 be the 3-dimensional unit sphere in $\mathbb{R}^4$, equipped with the standard Lie group structure, and let x be a non-zero element of the de Rham cohomology $H^3(S^3, \mathbb{R})$. Prove that there does not exist a Lie group isomorphism $f : S^3 \to S^3$ such that $f^*(x) = -x$.

3.

设 $\mathbb{C}P^2$ 为复射影平面, L_1, L_2, L_3 为其中三条复射影直线，满足 $L_1 \cap L_2 \cap L_3 = \emptyset$. 取 L_1, L_2, L_3 各自的紧致管状邻域，使得它们的并是一个 (实)4 维带边紧致流形 W，边界记为 $M = \partial W$, 并且要求 $W \setminus (L_1 \cup L_2 \cup L_3)$ 同胚于 $M \times [0, 1)$. 请计算 M 的 $\mathbb{Z}$ 系数同调群.

Let $\mathbb{C}P^2$ be the complex projective plane, and L_1, L_2, L_3 be three complex projective lines such that $L_1 \cap L_2 \cap L_3 = \emptyset$. The union of some compact tubular neighborhoods of L_1, L_2, L_3 is a compact (real) 4–dimensional manifold W with boundary $M = \partial W$, such that $W \setminus (L_1 \cup L_2 \cup L_3)$ is homeomorphic to $M \times [0, 1)$. Compute the homology of M with coefficients in $\mathbb{Z}$.

4.

设 (M, g) 是 n 维黎曼流形 $(n \geq 3)$，截面曲率 $K \geq 0$. 设 $\gamma(t)$ 为测地线, $t \in [0, T)$, 其中 t 是弧长参数. 假设 J_1, ..., J_{n-1} 是沿 γ 的 Jacobi 场, 都垂直于 γ', 且在 γ 每一点都线性无关。假设对任何 i, j 都满足

$$(J_i'(0), J_j(0)) = (J_i(0), J_j'(0)),$$

其中 J_i' 表示 J_i 沿 γ' 方向的协变导数。证明：对每个 $i = 1, \ldots, n-1$, 和 $0 < s < t < T$, 都有 $\frac{|J_i(s)|_g}{s} \geq \frac{|J_i(t)|_g}{t}$.

Let (M, g) be a n-dimensional Riemannian manifold $(n \geq 3)$ with sectional curvature $K \geq 0$. Let $\gamma(t)$ be a geodesic, $t \in [0, T)$, where t is the arc-length parameter. Assume $J_1, \ldots, J_{n-1}$ are Jacobi vector fields along γ, all orthogonal to $\gamma'(t)$ and linearly independent at any point of γ. Assume further that

$$(J_i'(0), J_j(0)) = (J_i(0), J_j'(0))$$

for any i, j, where J_i' means covariant derivative of J_i with respect to γ'. Prove that for every $i = 1, \ldots, n-1$ and any $0 < s < t < T$, we have $\frac{|J_i(s)|_g}{s} \geq \frac{|J_i(t)|_g}{t}$.

5.

设 M 是 5 维紧致光滑流形 $SO(3)\times T^2$, 其中 T^2 为一个 2 维环面.

(1) 是否存在 M 上的光滑黎曼度量 g 使得 Ricci 曲率严格为正?

(2) 是否存在 M 上的光滑黎曼度量 g 使得 $Ric\equiv 0$?

如果存在请给出具体的例子，如果不存在请给出证明.

Let M be the 5-dimensional smooth compact manifold $SO(3)\times T^2$, where T^2 is a 2-dimensional torus.

(1) Is there any smooth Riemannian metric g on M with strictly positive Ricci curvature?

(2) Is there any smooth Riemannian metric g on M with $Ric\equiv 0$?

Write down concrete examples if they exist and give your proof if they do not exist.

四、应用与计算数学 Applied&Computational Mathematics

1.

一个简单图 G 称为“漂亮的”, 如果它的任意两个相邻顶点的度数不同. 对任意 $n\geq 2$, 定义 $f(n)$ 为 n 阶“漂亮的”简单图的边数的最大值. 求满足

$$\lim_{n\to\infty}\frac{\binom{n}{2}-f(n)}{n^a}=b$$

的实数 a,b（$b\neq 0$）.

A simple graph G is called beautiful if any two adjacent vertices have different degrees. For any $n\geq 2$, define $f(n)$ as the maximum number of edges in a beautiful graph with n vertices. Find the real numbers a,b $(b\neq 0)$ satisfying that

$$\lim_{n\to\infty}\frac{\binom{n}{2}-f(n)}{n^a}=b.$$

2.

令 a 和 b 为两个正整数, 在一个不透明的袋子里放了 a 个红球和 b 个蓝球. 红球和蓝球除了颜色以外的其它特征相同, 只能通过颜色来分辨. 小明进行如下的游戏: 每一轮她从袋子里随机抽取一个球, 如果这个球是蓝球, 那么游戏结束；如果是红球, 那么她将该球放回袋子并再加放一个红球到袋子之中（这样袋子中的红球增多了一个）. 令 $E_{a,b}$ 为游戏总轮数的期望.

(1) 当 a 与 b 为何值时, 期望 $E_{a,b}$ 取有限值?

(2) 将 $E_{a,b}$ 表示为 a 与 b 的函数.

(3) 假设小明知道袋中的总球数 N 但不知道 a 与 b 的值. 她先验地认为 a 在集合 $\{1,\dots,N-1\}$ 中均匀分布. 在第几轮抽到红球的情况下她可以有 90% 的把握猜测 $E_{a,b}$ 取无穷值?

Let a and b be two strictly positive integers. Given an obscure bag containing a red balls and b blue balls which only differ in color, Alice plays the following game. In each round she picks randomly a ball in the bag. If the ball is blue, then the game terminates; otherwise she puts the ball back and adds another red ball in the bag (hence the number of red balls in the bag increases by 1). We denote by $E_{a,b}$ the expectation of the number of total rounds of the game.

(1) For which values of a and b the expectation $E_{a,b}$ is finite?

(2) Determine the value of $E_{a,b}$ as a function of a and b.

(3) Assume that Alice knows the total number N of balls but does not know the values of a and b. She estimates a priori that the value of a uniformly distributes in $\{1,\dots,N-1\}$. At the end of which round (that she picks a red ball) she could guess with a certainty of 90% that $E_{a,b}$ would be infinite?

3.

给定 n 个正实数 $a_1,\cdots,a_n$，假设它们满足 $a_1^2+\cdots+a_n^2=1$ 和 $a_1+\cdots+a_n=a$. 证明存在一种挑选系数 $\epsilon_1,\cdots\epsilon_n$ 的方法，使得每个系数 ϵ_i 均为 1 或 -1，并且

$$|\epsilon_1 a_1+\cdots+\epsilon_n a_n|\le \frac{1}{a}.$$

比如当 $n=5$ 且 $a_1=\cdots=a_5=1/\sqrt{5}$ 时 $a=\sqrt{5}$，可以取 $\epsilon_1=\epsilon_2=\epsilon_3=1,\epsilon_4=\epsilon_5=-1$，这样 $|\epsilon_1 a_1+\cdots+\epsilon_5 a_5|=1/\sqrt{5}=1/a$.

Given n positive real numbers $a_1, \cdots, a_n$ such that they form a unit vector, i.e. $a_1^2 + \cdots + a_n^2 = 1$. Let $a = a_1 + \cdots + a_n$. Show that there exist coefficients $\epsilon_i \in \{-1, 1\}$ for $i = 1, \cdots, n$, such that

$$|\epsilon_1 a_1 + \cdots + \epsilon_n a_n| \leq \frac{1}{a}.$$

For example, if $n = 5$ and $a_1 = \cdots = a_5 = 1/\sqrt{5}$, then $a = \sqrt{5}$, and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, $\epsilon_4 = \epsilon_5 = -1$ gives $\epsilon_1 a_1 + \cdots + \epsilon_5 a_5 = 1/\sqrt{5}$.

4.

在分子动力学中, 人们常使用 overdamped Langevin equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sqrt{2\beta^{-1}}\boldsymbol{\eta}$$

来采样 Boltzmann 分布 $\rho_\beta(\boldsymbol{x}) = Z_\beta^{-1} e^{-\beta V(\boldsymbol{x})}$, 这里 $\boldsymbol{x} \in \mathbb{R}^{3n}$, $\beta = \frac{1}{k_B T} > 0$, k_B 是 Boltzmann 常数, T 是温度, $\boldsymbol{f}(\boldsymbol{x}) = -\nabla V(\boldsymbol{x})$ 是由势函数 $V(\boldsymbol{x})$ 决定的作用力, $\boldsymbol{\eta}$ 是一个 3n 维的白噪声, 而 $Z_\beta = \int_{\mathbb{R}^{3n}} e^{-\beta V(\boldsymbol{x})} d\boldsymbol{x}$ 是归一化常数. 考虑如下的两条耦合的采样轨道,

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{f}(\boldsymbol{x}_1) + \sqrt{2\beta_1^{-1}(t)}\boldsymbol{\eta}_1 \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x}_2) + \sqrt{2\beta_2^{-1}(t)}\boldsymbol{\eta}_2, \end{cases} \tag{1}$$

其中 $\beta_1(t)$ 和 $\beta_2(t)$ 会交替地取值 $\beta > 0$ 和 $\bar{\beta} > 0$. 例如, 可选 $\bar{\beta} < \beta$ （即 $\bar{\beta}^{-1} > \beta^{-1}$）使得 $\bar{\beta}$ 对应的温度高于原系统的温度以提高采样效率. 按照频率 ν， $\beta_1(t)$ 和 $\beta_2(t)$ 会尝试互换取值, 如果互换是尝试从 $(\beta_1, \beta_2) = (\beta, \bar{\beta})$ 变成 $(\beta_1, \beta_2) = (\bar{\beta}, \beta)$, 那么这种互换的接受概率为

$$\min\left(\frac{\rho_{\bar{\beta}}(\boldsymbol{x}_1)\,\rho_\beta(\boldsymbol{x}_2)}{\rho_\beta(\boldsymbol{x}_1)\,\rho_{\bar{\beta}}(\boldsymbol{x}_2)}, 1\right),$$

而另外一种互换的接受概率类似可得. 写出（无需证明）当频率 $\nu \to \infty$ 时(1)的极限方程, 我们称之为系统（A）. 写出另一个 $\boldsymbol{x}_1$ 和 $\boldsymbol{x}_2$ 满足的随机动力方程, 我们称之为系统（B）, 使得系统（B）中只含有常系数的噪声项, 且系统（A）和系统（B）对应一样的不变分布.

In molecular dynamics, the overdamped Langevin equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sqrt{2\beta^{-1}}\boldsymbol{\eta}$$

is used to sample the Boltzmann distribution $\rho_\beta(\boldsymbol{x}) = Z_\beta^{-1} e^{-\beta V(\boldsymbol{x})}$, where $\boldsymbol{x} \in \mathbb{R}^{3n}$, $\beta = \frac{1}{k_B T} > 0$, k_B is the Boltzmann constant, T denotes the temperature, $\boldsymbol{f}(\boldsymbol{x}) = -\nabla V(\boldsymbol{x})$ is the force associated with the potential $V(\boldsymbol{x})$, $\boldsymbol{\eta}$ is a $3n$-dimensional white noise and $Z_\beta = \int_{\mathbb{R}^{3n}} e^{-\beta V(\boldsymbol{x})} d\boldsymbol{x}$. In a variant model, we consider two coupled sampling trajectories

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{f}(\boldsymbol{x}_1) + \sqrt{2\beta_1^{-1}(t)}\boldsymbol{\eta}_1 \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x}_2) + \sqrt{2\beta_2^{-1}(t)}\boldsymbol{\eta}_2, \end{cases} \tag{1}$$

where $\beta_1(t)$ and $\beta_2(t)$ alternatively swap between two values $\beta > 0$ and $\bar{\beta} > 0$. For example, we may take $\bar{\beta} < \beta$ (so that $\bar{\beta}^{-1} > \beta^{-1}$), and thus the sampling efficiency is improved because $\bar{\beta}$ corresponds to a higher temperature. These swaps are attempted with frequency ν, and the ones from $(\beta_1, \beta_2) = (\beta, \bar{\beta})$, $(\beta_1, \beta_2) = (\bar{\beta}, \beta)$ are accepted with probability

$$\min\left(\frac{\rho_{\bar{\beta}}(\boldsymbol{x}_1)\rho_\beta(\boldsymbol{x}_2)}{\rho_\beta(\boldsymbol{x}_1)\rho_{\bar{\beta}}(\boldsymbol{x}_2)}, 1\right),$$

and similarly for the ones from $(\beta_1, \beta_2) = (\bar{\beta}, \beta)$ to $(\beta_1, \beta_2) = (\beta, \bar{\beta})$. Write out (without proof) the limit equations of (1) when $\nu \to \infty$, which we name System (A). Find another dynamics for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, called System (B), such that System (B) contains only constant-coefficient noise terms, and System (A) and System (B) share the same invariant measure.

5.

假设 $x_1, \ldots, x_n$ 是一组相异实数, $y_1, \ldots, y_n$ 是另一组相异实数, 并且对于每个 $i = 1, \ldots, n$ 都有 $y_i \geq x_i$. 一个单向运输 T 是一个从 $\{x_1, \ldots, x_n\}$ 到 $\{y_1, \ldots, y_n\}$ 的一一映射，并且满足对于每个 $i = 1, \ldots, n$ 都有 $T(x_i) \geq x_i$. (例如，对每个 $i = 1, \ldots, n$ 使 $T(x_i) = y_i$ 就定义了一个单向运输.) T 的运输成本定义为

$$\sum_{i=1}^{n} (T(x_i) - x_i)^2.$$

(1) 找到或描述分别最小化和最大化运输成本的两个单向运输, 并证明它们的最优性.

(2) 假设 $(x_n)_{n=1,2,\ldots}$ 是一列来自标准正态分布的独立同分布序列，并且 $y_i = x_i + 1$, $i = 1, 2, \ldots$. 令 T_n^* 为从 $X_n := \{x_1, \ldots, x_n\}$ 到 $Y_n := \{y_1, \ldots, y_n\}$ 的最大化运输成本的单向运输. 计算以下随机量

$$\frac{\#\{x \in X_n : x \leq 0,\ T_n^*(x) \leq 1\}}{n}$$

当 $n \to \infty$ 时的极限 (在几乎必然 (a.s.) 意义下), 其中 # 表示集合的势.

Let $x_1, \ldots, x_n$ be n distinct real numbers and $y_1, \ldots, y_n$ are also distinct such that $y_i \geq x_i$ for $i = 1, \ldots, n$. A directional transport T is a bijection from $\{x_1, \ldots, x_n\}$ to $\{y_1, \ldots, y_n\}$ such that $T(x_i) \geq x_i$ for $i = 1, \ldots, n$. (For instance, letting $T(x_i) = y_i$ for $i = 1, \ldots, n$ defines a directional transport.) The transportation cost of T is

$$\sum_{i=1}^{n} (T(x_i) - x_i)^2.$$

(1) Find or describe two directional transports which, respectively, minimize and maximize the transportation cost, and prove their optimality.

(2) Suppose that $(x_n)_{n=1,2,\ldots}$ is an iid sequence from the standard normal distribution and $y_i = x_i + 1$, $i = 1, 2, \ldots$. Let T_n^* be the directional transport with maximum transportation cost from $X_n := \{x_1, \ldots, x_n\}$ to $Y_n := \{y_1, \ldots, y_n\}$. Find the limit (in the almost sure sense) of the random quantity

$$\frac{\#\{x \in X_n : x \leq 0,\ T_n^*(x) \leq 1\}}{n}$$

as $n \to \infty$, where $\#$ is the cardinality of a set.